Fractional quantum Hall effect on the two-sphere: a matrix model proposal

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Abstract

We present a Chern-Simons matrix model describing the fractional quantum Hall effect on the two-sphere. We demonstrate the equivalence of our proposal to particular restrictions of the Calogero-Sutherland model, reproduce the quantum states and filling fraction and show the compatibility of our result with the Haldane spherical wavefunctions.

1 Introduction

In paper [1], Susskind proposed to describe the fractional quantum Hall fluid using non-commutative Chern-Simons field theory. (For some recent reviews of noncommutative field theory see [2–5].) In [1], the inverse filling fraction is identified with the Chern-Simons level, and the coordinates of electrons are elevated to matrices which are essentially the spatial components of the noncommutative gauge field. Just as in the ordinary Chern-Simons, the spatial components of the gauge field have nontrivial commutation relations and as a consequence the electron coordinates are automatically projected to the lowest Landau level. The model [1] describes an infinite number of electrons and, modulo gauge invariance, has only one state.

Since the original proposal [1], significant work has been done extending the model and relating it to previous descriptions of the fractional quantum Hall effect. A matrix model describing finite size Hall droplets and supporting boundary excitations was introduced in [6]. The model naturally contains quasiholes and one can derive their quantized charge and the quantization of the inverse filling fraction. The relation between the states of the model [11] and the Laughlin states was further explored in [7–10]. See also [11] for an extension of the model describing the Hall fluid on a cylinder and [12] for an attempt to describe a multi-layered Hall fluid.

The construction of the corresponding matrix models on compact spaces, however, presents special challenges. In particular, no Chern-Simons like action exists for the two-sphere. Alternative approaches have been suggested in [13] where two complex matrices were used to parametrize the coordinates of the electrons and in [14] where a fermionic matrix model was introduced to describe a Hall fluid at filling fraction one. See also [15] for further discussions. The lack of a Chern-Simons type action is surprising, since Laughlin's treatment [16] of the fractional quantum Hall effect was easily extended by Haldane [17] to the two-sphere. The matrix model describing this situation is expected to reproduce features of the "fuzzy sphere", at least for the fully filled case. (For early work on fuzzy spheres and some further discussions see [18–24].)

In this paper we shall try to rectify this situation by proposing such a matrix model using stereographic coordinates on the sphere and a boundary field similar to the one in the finite planar model of [6]. Furthermore, we show, using two different natural changes of variables,

that our model is equivalent to either a Calogero [25] or a Sutherland [26] integrable model. Some restrictions exist on the values of the integrals of motion but these are consistent with the dynamics, even when we perturb it by reasonable hamiltonians. We also show that once these restrictions are imposed, the Calogero or Sutherland models acquire a new $SO(3)$ symmetry corresponding to three dimensional rotations of the spherical Hall fluid.

The plan of the paper is as follows. In Section 2 we present the action for a charged (lagrangian) fluid on a two-sphere in a strong magnetic field. In Section 3 we introduce a matrix model whose classical limit is the fluid model. We discuss ordering issues and show that the model admits the fuzzy sphere as a solution. In Section 4 we present our second, finite model which incorporates a boundary field. In Section 5 we show using the equivalence of the model to either a Calogero or a Sutherland model with some restrictions on the values of the integrals of motions. In Section 6 we perform the quantization of the model and compare its states with the corresponding Haldane states on the sphere. Finally, we conclude with a set of comments and open questions in Section 7.

2 Lagrangian fluid model

As discussed in [27], and more recently reviewed in [1], one can describe the Hall fluid using a lagrangian fluid model. The electrons of the fluid are described in the Lagrange formulation by their spatial coordinates $x_a(y_1, y_2)$ which are functions of the particle-fixed coordinates y_a ($a = 1, 2$ for the Hall system). Assuming that the body-fixed coordinates represent a reference state of constant particle density ρ_0 and charge density $e\rho_0$, the lagrangian of the model is given by

$$\mathcal{L} = eB\rho_0 \int d^2y \left[\frac{1}{2} \epsilon^{ab} x_a \dot{x}_b + \frac{A_0}{2\pi\rho_0} \left(\{x_1, x_2\} - 1 \right) \right] , \tag{1}$$

where $\{x_1, x_2\}$ are the Poisson brackets on the manifold y_1, y_2:

$$\{x_1, x_2\} = \epsilon^{ab} \partial_a x_1 \partial_b x_2 = \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2} \frac{\partial x_2}{\partial y_1} . \tag{2}$$

The first term in $\mathcal{L}$ is the coupling of the charged fluid to a constant magnetic field B. A constant density for the fluid equal to ρ_0 is enforced by the lagrange multiplier A_0. For more details and the motivation for this model see the discussion in [1].

We use the same approach to treat the Hall fluid on a two-sphere. The gauge field for a uniform magnetic field of strength B on a two-sphere can be written in stereographic coordinates (up to a gauge transformation) as

$$i2BR^2(1+z\bar{z})^{-1}\bar{z}dz \ . \tag{3}$$

(From now on we put $e = 1$ for simplicity.) Near the north pole of the sphere the stereographic coordinate z becomes $\sim (x+iy)/2R$ in terms of local Cartesian coordinates, while $z = \infty$ represents the south pole and the gauge potential has a Dirac-string singularity there.

The lagrangian description of the Hall fluid is given by a map $z(\sigma, \bar{\sigma})$ where both z and σ are stereographic coordinates on the two-sphere. We can still pick the lagrangian parametrization $(\sigma, \bar{\sigma})$ such that it corresponds to a reference configuration of uniform particle density ρ_0 in σ space

$$dN = \rho_0 \frac{2R^2}{(1+\sigma\bar{\sigma})^2} d\sigma d\bar{\sigma} \ . \tag{4}$$

The coupling of the charged fluid to the magnetic field (3) is given by

$$\mathcal{L}' = i2BR^2 \int \rho_0 \frac{2R^2 d\sigma d\bar{\sigma}}{(1+\sigma\bar{\sigma})^2} (1+z\bar{z})^{-1}\dot{z}\bar{z} \ . \tag{5}$$

As in [1] we consider a limit where the coupling to the magnetic field dominates the standard quadratic kinetic and potential terms, and the latter terms will be ignored in what follows (a classical lowest Landau level reduction [28]).

As in the planar case, we want to restrict the model to configurations of uniform density. Then the infinitesimal particle number in z space must be given by

$$dN = \rho_0 \frac{2R^2}{(1+z\bar{z})^2} dz d\bar{z} \ , \tag{6}$$

with the same ρ_0 as in (4). Thus the map $z(\sigma, \bar{\sigma})$ must be area preserving, and we have

$$(1+z\bar{z})^{-2} \left(\frac{\partial z}{\partial \sigma} \frac{\partial \bar{z}}{\partial \bar{\sigma}} - \frac{\partial z}{\partial \bar{\sigma}} \frac{\partial \bar{z}}{\partial \sigma} \right) = (1+\sigma\bar{\sigma})^{-2} \ . \tag{7}$$

We can rewrite this using the Poisson bracket on the sphere

$$i\{F, G\} = (1+\sigma\bar{\sigma})^2 \left(\frac{\partial F}{\partial \sigma} \frac{\partial G}{\partial \bar{\sigma}} - \frac{\partial F}{\partial \bar{\sigma}} \frac{\partial G}{\partial \sigma} \right) \ . \tag{8}$$

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Note that the Poisson tensor is just the inverse of the symplectic (area) 2-form in stereographic coordinates. Using (8) we write the constraint (7) as

$$(1 + z\bar{z})^{-2}\{z, \bar{z}\} + i = 0 \ . \tag{9}$$

We can obtain (9) directly from a lagrangian by introducing a multiplier A_0

$$\mathcal{L} = i2BR^2 \int \rho_0 \frac{2R^2 d\sigma d\bar{\sigma}}{(1 + \sigma\bar{\sigma})^2} \left\{ (1 + z\bar{z})^{-1}\dot{z}\bar{z} + \frac{A_0}{2\pi\rho_0} \left[(1 + z\bar{z})^{-2}\{z, \bar{z}\} + i \right] \right\} \ . \tag{10}$$

As the notation suggests, A_0 can be interpreted as a gauge field. Indeed, defining the covariant derivative

$$D_0 z = \dot{z} + \theta\{A_0, z\} \ , \tag{11}$$

where $\theta = (2\pi\rho_0)^{-1}$, we can combine the first two terms in the lagrangian and (up to a total derivative) we have

$$\mathcal{L} = i2eBR^2 \int \rho_0 \frac{2R^2 d\sigma d\bar{\sigma}}{(1 + \sigma\bar{\sigma})^2} \left\{ (1 + z\bar{z})^{-1} D_0 z\bar{z} + \frac{iA_0}{2\pi\rho_0} \right\} \ . \tag{12}$$

The gauge group corresponding to A_0 is the infinite dimensional group of area preserving diffeomorphisms.

3   Noncommutative model

The noncommutative version (matrix model) of the fluid model on the sphere is obtained by promoting the field $z(\sigma, \bar{\sigma})$ into a matrix z and turning the two-dimensional integration into a matrix trace

$$\int \frac{2R^2 d\sigma d\bar{\sigma}}{(1 + \sigma\bar{\sigma})^2} \rightarrow 2\pi\theta \text{Tr} \ . \tag{13}$$

The dimension of the matrix N will represent the number of particles.

In writing the appropriate action, we face issues of matrix ordering, since the matrices z, $z^\dagger$ and $\dot{z}$ do not commute. We shall use a lagrangian L with an ordering in which z and $z^\dagger$ alternate; specifically the kinetic term is given by:

$$\mathcal{L}'(z, z^\dagger) = i2BR^2 \text{Tr} \left[(1 + zz^\dagger)^{-1}\dot{z}z^\dagger \right] \ . \tag{14}$$

The same ordering also appears in the kinetic terms of certain nonlinear sigma models discussed in [29,30]. Such an ordering ensures that the above lagrangian properly describes a model on the sphere. As in the single-particle (scalar) case, this requires that the canonical structure obtained by the above lagrangian be invariant under a set of $SO(3)$ rotations. Such transformations are a subgroup of the full group $SL(2,\mathbb{C})$ of modular transformations defined, as in the scalar case, by

$$z' = (Az + B)(Cz + D)^{-1} , \tag{15}$$

with A, B, C, D complex numbers satisfying $AD - CB = 1$. Specifying $SO(3)$, or equivalently $SU(2)/\mathbb{Z}_2$ transformations, amounts to choosing a unitary matrix of the above coefficients:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\dagger} = \mathbb{I} . \tag{16}$$

Note also that an overall phase in this matrix is irrelevant in the transformations (15). Then we can show that the above lagrangian changes by a total derivative,

$$\mathcal{L}'(z', z'^{\dagger}) = \mathcal{L}'(z, z^{\dagger}) + \frac{d}{dt} \left\{ i2BR^2 \, \mathrm{Tr} \left[\ln(Cz + D) \right] \right\} \tag{17}$$

and therefore the canonical form remains invariant. Note that the chosen ordering is crucial for this property[a].

In infinitesimal form, the relations (16) imply

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \approx \mathbb{I} + \frac{i}{2} \begin{pmatrix} a & b \\ \bar{b} & -a \end{pmatrix} , \tag{18}$$

with $\bar{a} = a$. This corresponds to the transformation

$$\delta z = i \left(az + \frac{1}{2}b - \frac{1}{2}\bar{b}z^2 \right) \tag{19}$$

representing infinitesimal rotations. By Noether's theorem, the conserved charges corresponding to the above rotations are

$$\begin{cases} J_3 &=& BR^2 \, \mathrm{Tr} \left[(1 + zz^{\dagger})^{-1} + (1 + z^{\dagger}z)^{-1} - 1 \right] , \\ J_+ &=& 2BR^2 \, \mathrm{Tr} \left[(1 + zz^{\dagger})^{-1}z \right] , \\ J_- &=& 2BR^2 \, \mathrm{Tr} \left[z^{\dagger}(1 + zz^{\dagger})^{-1} \right] . \end{cases} \tag{20}$$

[a]Stereographic coordinates have been used previously to study fuzzy spheres in [31,32]. However, these papers use a slightly different definition of z in terms of "Cartesian coordinates" than is used here. As a consequence z does not transform by fractional transformations.

These are the generators of the transformation (18) and consequently their Poisson brackets satisfy the $SU(2)$ rotation algebra

$$
\begin{aligned}
\{J_\pm, J_3\} &= \pm i\, J_\pm\ , \\
\{J_-, J_+\} &= i2\, J_3\ .
\end{aligned}
\tag{21}
$$

This result can be directly verified by using the canonical Poisson brackets implied by the lagrangian, from which we read off the Poisson structure

$$
\{z_{a_1 b_1}, [z^\dagger (1 + z z^\dagger)^{-1}]_{a_2 b_2}\} = \frac{1}{i2BR^2} \delta_{a_1 b_2} \delta_{a_2 b_1}\ .
\tag{22}
$$

The above can be written in a more convenient index-independent notation, denoting by 1 and 2 the spaces in which indices (a_1, b_1) and (a_2, b_2) act respectively, and by T_{12} the operator exchanging the two spaces. In this notation we have

$$
\{z_1, z_2^\dagger (1_2 + z_2 z_2^\dagger)^{-1}\} = \frac{1}{i2BR^2} T_{12}\ .
\tag{23}
$$

The Poisson brackets of z and $z^\dagger$ derive from above as:

$$
\{z_1, z_2^\dagger\} = \frac{1}{i2BR^2}(1_1 + z_1 z_1^\dagger)\, T_{12}\, (1_1 + z_1^\dagger z_1)\ .
\tag{24}
$$

Using (24) we can show that the $SU(2)$ Poisson brackets (21) hold. (More details of the derivation are given in the appendix.)

The ordering of the matrices inside the trace in the definition of $J_\pm$ and J_3 is essentially fixed by the alternating z and $z^\dagger$ rule, up to cyclic permutations. The first two terms in the definition of J_3 are actually equal inside the trace. The particular ordering was used for the following reason: the matrices

$$
\left\{
\begin{aligned}
X_3 &= R[(1 + z z^\dagger)^{-1} + (1 + z^\dagger z)^{-1} - 1]\ , \\
X_+ &= 2R(1 + z z^\dagger)^{-1} z\ , \\
X_- &= 2R z^\dagger (1 + z z^\dagger)^{-1}\ ,
\end{aligned}
\right.
\tag{25}
$$

transform to each other as a vector under the transformation (19), as can be directly checked. They are, therefore, prime candidates for the (Cartesian) matrix coordinates on a sphere. This point will be relevant in what follows.

The matrix model presented above describes a "fuzzy fluid" consisting of N particles on the sphere in the presence of a constant magnetic field. We would also like to incorporate

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the incompressibility condition that the density on the sphere of this fluid remains constant. Just as in the planar case, this can be implemented by gauging the time derivative

$$\dot{z} \rightarrow \dot{z} - i[A_0, z] \,, \tag{26}$$

with A_o a hermitian matrix gauge field, and adding the term $B\theta \text{Tr} A_0$ in the lagrangian. The Gauss law obtained from varying A_0 reads

$$[z, z^\dagger(1 + zz^\dagger)^{-1}] \equiv (1 + z^\dagger z)^{-1} - (1 + zz^\dagger)^{-1} = \frac{\theta}{2R^2} \,. \tag{27}$$

This is the noncommutative (matrix) analog of the fluid condition that fixes the density on the sphere to

$$\rho_0 = \frac{1}{2\pi\theta} \tag{28}$$

(given that $z \sim x/2R$).

The above commutation relations (27) are very suggestive. Consider the three "coordinate" matrices $X_\pm$ and X_3 as defined in (25). We can show that the commutation relations (27) are formally equivalent to $SU(2)$ commutation relations among the coordinates up to a factor of θ/R:

$$[X_3, X_\pm] = \pm\frac{\theta}{R}X_\pm \,, \quad [X_+, X_-] = \frac{\theta}{R}2X_3 \,. \tag{29}$$

Moreover, the sum of squares of the above three matrices is

$$
\begin{aligned}
X_+ X_- + X_- X_+ + 2X_3^2 &= 2R^2 \left\{ 1 - [(1 + z^\dagger z)^{-1} - (1 + zz^\dagger)^{-1}]^2 \right\} \\
&= 2R^2 \left[1 - \left(\frac{\theta}{2R^2}\right)^2 \right] \,,
\end{aligned}
\tag{30}
$$

the last equality being valid upon use of the Gauss law. We see that the above matrices become coordinates of a fuzzy sphere of radius R and noncommutativity parameter θ. Identifying (30) with (θ^2/R^2 times) the quadratic Casimir $j(j+1)$ also implies a quantization of θ according to

$$\theta = \frac{R^2}{j + \frac{1}{2}} = \frac{2R^2}{N} \tag{31}$$

for $N = 2j + 1$, as is standard in noncommutative spheres. So this would correspond to a completely filled sphere, in analogy to the completely filled plane of the standard (planar) noncommutative Chern-Simons model.

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Unfortunately, the matrix relation (27) is inconsistent. By tracing both sides of the Gauss law we see that, just as in the planar case, no finite-dimensional matrices can satisfy it. Unlike the planar case, however, it admits no infinite-dimensional representations either. To see this, rewrite the above commutator as

$$[z, z^\dagger z(1 + z^\dagger z)^{-1}] = \frac{\theta}{2R^2}z \ . \tag{32}$$

So z and $z^\dagger$ act as lowering and raising operators, respectively, for both $zz^\dagger(1 + zz^\dagger)^{-1}$ and $z^\dagger z(1 + z^\dagger z)^{-1}$. (Assuming $\theta > 0$; else the role of z and $z^\dagger$ is reversed.) But the spectrum of these last two operators is bounded from above by 1; therefore, there must be a highest state, annihilated by $z^\dagger$. On this state, however, the relation (27) implies that $z^\dagger z(1 + z^\dagger z)^{-1}$ has a negative eigenvalue, which is also inconsistent.

Nevertheless, it is remarkable that the fuzzy sphere algebra (29) can be formally extracted from the commutation relations (27). Note further that we should not expect the matrix elements of z and $z^\dagger$ to be finite. In the large N limit such finite matrices would map to a nonsingular function $z(\sigma, \bar\sigma)$ from S^2 to $\mathbb{C}$. However, for the fully filling Hall fluid $z(\sigma, \bar\sigma)$ is always singular at the south pole. Strictly speaking z should be considered as a local coordinate map of a functions from S^2 to S^2 of winding number one. Just as for the classical case it is useful to work with singular functions, in the matrix case we should work with the algebra generated by z and $z^\dagger$ modulo the relations (27). While this algebra, as shown above, has no unitary representations, it contains a subalgebra (generated by $J_\pm, J_2$) that has finite dimensional unitary representations.

This interpretation is useful if we are willing to work with only the equation of motion. For the fully filled sphere this is perhaps enough, since modulo gauge transformations there exists only one state. For partially filled states, however, and for the purposes of quantization, this description fails. We conclude that the naïve gauged matrix model on the sphere cannot stand, unlike the planar case. This is due to the compactness of space: a Laughlin (or rather Haldane) state on the sphere has a finite number of particles and would require a finite-dimensional matrix, which is inconsistent with the commutation relation (27).

4 Finite matrix model

In what follows we introduce a modified model containing a boundary field, just as in the planar or cylindrical case. As a result the model can describe Hall fluid states that are only partially filling the two-sphere with the matrices z and $z^\dagger$ having finite dimensional representations.

The full spherical boundary matrix model lagrangian reads

$$
\begin{aligned}
\mathcal{L}(z, z^\dagger) &= iB \operatorname{Tr} \left[2R^2(1+zz^\dagger)^{-1}\dot{z}z^\dagger - iA_o(2R^2[z, z^\dagger(1+zz^\dagger)^{-1}] - \theta) \right] \\
&+ \Psi^\dagger(i\dot{\Psi} + A_o\Psi) ,
\end{aligned}
\tag{33}
$$

with Ψ a complex column N-vector. In the limit $z \ll 1$ this reduces to the planar finite quantum Hall matrix model, upon identifying $X + iY = 2Rz$. This model still admits the transformation $z' = (Az + B)(Cz + D)^{-1}$ as a symmetry, with A_0 and Ψ transforming trivially, and thus properly describes a spherical system. A hamiltonian term $-\operatorname{Tr}V(z, z^\dagger)$ can also be added, just as in the planar case, representing a potential that would tend to concentrate the droplet and assign different energies to different states.

The classical analysis of the dynamics of the model parallels the one of the planar case. The Gauss law constraint, now, reads

$$
2BR^2[z, z^\dagger(1+zz^\dagger)^{-1}] + \Psi\Psi^\dagger = B\theta .
\tag{34}
$$

This equation admits finite-dimensional representations, under some conditions for R, B and θ. We shall demonstrate the solution corresponding to a circular droplet centered around the north pole, a sort of ground state configuration.

Tracing (34) we obtain

$$
\Psi^\dagger\Psi = BN\theta .
\tag{35}
$$

By a gauge rotation we can bring the N-vector Ψ to the form

$$
\Psi_n = \delta_{n,N}\sqrt{BN\theta} , \quad n = 1, \dots N .
\tag{36}
$$

In this basis we shall choose z and $z^\dagger$ to act as lowering and raising matrices

$$
z_{mn} = a_m\delta_{m+1,n} , \quad m, n = 1, \dots N
\tag{37}
$$

determined by the $N-1$ constants $a_1, \ldots a_{N-1}$ (a_N does not appear since $\delta_{N+1,n} = 0$). We have

$$(z^\dagger z)_{mn} = |a_{m-1}|^2 \delta_{mn} , \quad (zz^\dagger)_{mn} = |a_m|^2 \delta_{mn} , \tag{38}$$

with the convention $a_0 = a_N = 0$. Putting $b_n = |a_n|^2/(1 + |a_n|^2)$ the commutation relation (34) gives

$$b_n - b_{n-1} = \frac{\theta}{2R^2}(1 - N\delta_{n,N}) , \tag{39}$$

which admits as solution

$$b_n = n\frac{\theta}{2R^2} , \quad n < N, \quad b_N = 0 . \tag{40}$$

The "problematic" top state $n = N$, which was giving the inconsistency in the absence of the boundary field, is now consistent. Finally, the a_n can be determined from the b_n as

$$|a_n|^2 = \frac{\theta n}{2R^2 - \theta n} , \quad n < N , \quad a_N = 0 . \tag{41}$$

The phases of a_n are gauge degrees of freedom and can be chosen zero.

Consistency requires that the right-hand side above be non-negative. This imposes the constraint

$$\theta \leq \frac{2R^2}{N-1} . \tag{42}$$

To see what this means, we rewrite it as

$$2\pi\theta(N-1) \leq 4\pi R^2 . \tag{43}$$

The right hand side is the area of the sphere. $2\pi\theta$ is the "area quantum" occupied by each particle, or rather, the area excluded from occupation by the remaining particles. Thus, positioning N particles on the sphere requires an area of at least (N-1) such quanta.

The limiting case when $\theta(N-1) = 2R^2$ corresponds to positioning the first particle on the north pole, and positioning subsequent ones on "fuzzy rings" at a vertical intervals of $2R/(N-1)$, with the last particle barely squeezing at the south pole. This represents a fully filled sphere. Such a state is, actually, rotationally invariant, as can be seen from the fact that the generators $J_\pm, J_3$ vanish in this case, or from the fact that the $SO(3)$ transformations are equivalent to $SU(N)$ conjugations (gauge transformations) of z in this

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case. This is the classical noncommutative analog of the fully filled Laughlin-Haldane state on the sphere. (Note that one matrix element of z becomes singular in this case.)

The constraint (42) above differs from the one in (31) by a shift of $j + \frac{1}{2}$ to j. Such shifts are common in the identification of R in terms of j and θ for the fuzzy sphere. Heuristically we may say that each particle occupies an area of $2\pi\theta$ in the previous fuzzy sphere picture, while $2\pi\theta$ represents only an exclusion area in the present picture.

5 Equivalence to the Calogero-Sutherland particle model

The dynamics of the above gauged matrix model with boundary field is somewhat complicated to obtain, due to the nonlinear form of the action in terms of z. It can be recast in more familiar form, however, upon proper redefinition of variables. There are at least two different ways to do this, leading, respectively, to a correspondence with the well-known Calogero and Sutherland integrable systems.

To obtain the first correspondence, we define a new matrix w as

$$w = 2Rz(1 + z^\dagger z)^{-\frac{1}{2}} \ . \tag{44}$$

Since $z^\dagger z$ is a non-negative hermitian matrix, the square root above is defined in the usual way in terms of the positive square roots of the eigenvalues. In the scalar case the variable w would represent the "chord length" of the particle from the north pole, along with its azimuthal angle; the above is the proper matrix generalization. The inverse transformation reads

$$z = w(4R^2 - w^\dagger w)^{-\frac{1}{2}} \ . \tag{45}$$

The lagrangian in terms of the new variable assumes the form

$$\mathcal{L} = i\frac{B}{2} \operatorname{Tr} \left[w^\dagger \dot{w} - iA_0([w, w^\dagger] - 2\theta) \right] + \Psi^\dagger(i\dot{\Psi} + A_o \Psi) \ . \tag{46}$$

This is identical to the lagrangian of the planar Chern-Simons matrix model. The only remnant of the spherical topology is a constraint on w: from its definition, it is clear that $w^\dagger w$ cannot have eigenvalues greater that $2R$. That is, it must satisfy the constraint

$$w^\dagger w \leq 4R^2 \tag{47}$$

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($ww^\dagger$ has the same spectrum, apart, possibly, from zero modes and will obey the same constraint.) The solution of this model, then, will map to the solution of the corresponding truncation of the set of solutions of the planar model.

Addition of single-particle potentials amounts to the addition of a term $-\mathrm{Tr}V(z,z^\dagger)$ in the action, which will map to a corresponding planar potential $\tilde{V}(w,w^\dagger)$. Interestingly, the harmonic oscillator potential on the plane

$$\tilde{V} = \frac{1}{2}\omega^2 w^\dagger w \tag{48}$$

maps (up to irrelevant constants) to a constant electric field in the vertical direction on the sphere,

$$V = 2R^2\omega^2(1+z^\dagger z)^{-1} \sim -R^2\omega^2\cos\theta \tag{49}$$

corresponding to an electric field $E = \omega^2 R$. The above potential, as well as any potential that is a function of $w^\dagger w$ alone, conserves the matrix $w^\dagger w$ over time and thus will also preserve the constraint $w^\dagger w < 4R^2$ on its eigenvalues, once they are satisfied for the initial configuration. Arbitrary potentials $\tilde{V}$ in w, however, can produce motions that exit the constraint subspace. Such potentials always map to spherical potentials V with singular behavior at the south pole and can be eliminated as unphysical.

As is well established, the planar model can be mapped to the Calogero system of particles (for a review see [33, 34]). The details of the mapping have been presented before and will not be repeated here[b]. We just state the relevant results.

The eigenvalues of the hermitian part of the matrix w, $X = (w+w^\dagger)/2$, become particle coordinates, while the diagonal elements of the antihermitian part $P = B(w-w^\dagger)/(2i)$, in the basis where X is diagonal, become their conjugate momenta. With the harmonic potential (48) as the hamiltonian, the particle hamiltonian is the standard Calogero hamiltonian with a one-body harmonic oscillator potential and a two-body inverse-square potential with strength θ^2. The eigenvalues of the matrix $w^\dagger w$ are a set of classical "pseudo-energies" e_i, whose sum is the total energy. Higher conserved quantities of the model are written as

$$I_n = \mathrm{Tr}(w^\dagger w)^n = \sum_{i=1}^{N} e_i^n \quad, n = 1, 2, \ldots N \ . \tag{50}$$

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[b]For a direct mapping of planar lowest Landau level anyons and quantum Hall states to the Calogero system see [35, 36]

The constraint $[X, P] = iB\theta - i\Psi\Psi^\dagger$ implies that the distance between the e_i is at least 2θ. The ground state, corresponds to the Calogero particles at rest in their equilibrium position. It achieves the minimal values for the pseudo-energies, namely $0, 2\theta, 4\theta, \ldots, 2(N-1)\theta$. For the ground state to exist, it must satisfy the constraint $w^\dagger w \leq 4R^2$ so we obtain

$$(N - 1)\theta \leq 2R^2 \ . \tag{51}$$

This constraint, and the ground state, are the ones obtained in the previous section. Excited states correspond to any other Calogero particle configuration satisfying the constraints $e_i \leq 4R^2$.

The second way to reduce the model amounts to a different parametrization of the matrix z. Specifically, any matrix can be parametrized in terms of a "modulus" and "phase" part, as

$$z = hU \ , \tag{52}$$

where h is a hermitian matrix and U is a unitary matrix. They can be expressed as

$$h = (zz^\dagger)^{\frac{1}{2}} \ , \quad U = (zz^\dagger)^{-\frac{1}{2}} z \ . \tag{53}$$

A further redefinition, more suitable for our purposes, is

$$H = (h^2 - 1)(h^2 + 1)^{-1} = (zz^\dagger - 1)(zz^\dagger + 1)^{-1} \ . \tag{54}$$

In the scalar case the above variable would represent $(1/R$ times) the vertical coordinate of the particle and is closely related to the matrix X_3 defined before in (25) (up to ordering). The parametrization in terms of H and U, then, is the matrix version of the canonical parametrization of the sphere in terms of X_3 and the azimuthal angle ϕ.

In terms of the new variables the lagrangian becomes (up to total derivatives)

$$\mathcal{L} = iBR^2 \operatorname{Tr}\left[H\dot{U}U^{-1} - iA_0(H - U^{-1}HU - \theta/R^2)\right] + \Psi^\dagger(i\dot{\Psi} + A_o\Psi) \ . \tag{55}$$

This is identical to the lagrangian of the unitary matrix model describing cylindrical quantum Hall states. Again, the only remnant of the spherical topology is a constraint on H: from its definition, we see that its eigenvalues have to be in the range $[-1, 1]$. The solution of the model will, then, map to the solutions of the cylindrical model truncated to the

subspace satisfying the constraint. Similar remarks on the inclusion of one-body potentials apply as in the case of the previous mapping to the planar model.

The above matrix model is equivalent to the periodic modification of the Calogero model, known as the Sutherland model. Again, we shall not repeat the details but simply state the relevant facts.

The coordinates of the model can be taken as (the phase of) the eigenvalues of U, representing particles on a circle of unit area, while the diagonal elements of the matrix $P = BR^2 H$ in the basis where U is diagonal become particle momenta. The standard Sutherland model obtains upon choosing $\frac{1}{2}\mathrm{Tr}P^2$ as the hamiltonian, and includes two-body inverse-sine-square interactions with strength proportional to θ^2. The eigenvalues p_i of P become a set of "pseudo-momenta" whose sum is the total momentum p. The total momentum p, energy E and a tower of higher conserved quantities of the model are written as

$$I_n = \mathrm{Tr}P^n = \sum_{i=1}^{N} p_i^n \quad (I_1 = p, \;\; I_2 = 2E) \; . \tag{56}$$

The constraint $P - U^{-1}PU = B\theta - \Psi\Psi^\dagger$ implies that the distance between the eigenvalues p_i of P is at least $B\theta$. The ground state is when the particles are frozen to their equilibrium position and corresponds to the p_i assuming their minimal values around zero, namely $-B\theta(N-1)/2, \ldots B\theta(N-1)/2$. The eigenvalues of $H = P/BR^2$, then will range from $-\theta(N-1)/2R^2$ to $\theta(N-1)/2R^2$. For this to be compatible with the constraint $-1 \le H \le 1$ we must have

$$\theta\frac{N-1}{2R^2} \le 1 \; . \tag{57}$$

which leads, again, to the same constraint as before.

In conclusion, the spherical matrix model can be described in terms of the motion and degrees of freedom of either a Calogero or a Sutherland particle system, with the initial data restricted so that they satisfy the spherical constraint of the model. All solutions can be obtained in terms of the known classical solutions of these particle systems. This will also be useful in the quantization of the model.

6   Quantization

The full correspondence of the Chern-Simons matrix model to the fractional quantum Hall system is born out at the quantum level, where the quantization of the filling fraction and the charge of the quasiholes naturally emerge.

The quantization of the present model will proceed along similar lines as the quantization of the corresponding planar or cylindrical models. In fact, as was demonstrated in the previous section, these models are equivalent up to the special constraint that imposes the spherical topology.

To proceed, we choose the "planar" version of the model in terms of the matrix w. Upon quantization, w, $w^\dagger$ and Ψ, $\Psi^\dagger$ become matrices and vectors of oscillator ladder operators respectively:

$$[w_{a_a b_1}, w^\dagger_{a_2 b_2}] = \frac{2}{B}\delta_{a_1 b_2}\delta_{a_2 b_1} \ , \quad [\Psi_a, \Psi^\dagger_\beta] = \delta_{ab} \ . \tag{58}$$

The states of the model are generated by the repeated action of $w^\dagger$ and $\Psi^\dagger$ on the oscillator ground state $|0>$ annihilated by all matrix elements of w and Ψ. Gauge invariance and the Gauss law constraint impose restrictions on such states as well as the value of θ. The analysis is well known and we again state the results.

The Gauss law constraint quantum mechanically becomes a statement on the allowed representations of the generators of the conjugation symmetry of w, $w^\dagger$. Group theory constraints require the quantization condition [37,38,6]

$$\theta B = k \ , \quad k = 0, 1, 2, \dots \ . \tag{59}$$

This will be related to the quantization of the filling fraction.

The implementation of the spherical constraint (47) is more subtle and will lead to another quantization condition. Quantum mechanically it is unclear to what the eigenvalues of this matrix would correspond. One way to resolve the issue is to work, instead, with the invariants of the model I_n as given in (50). The restriction $e_i \leq 2R$ imposes corresponding restrictions on I_n.

Quantum mechanically, I_n are the corresponding commuting quantum integrals of the model. As is known from the matrix model, or its corresponding Calogero system, such

integrals can be written in terms of a set of quantum mechanical "pseudo-excitation num-bers" m_i, $i = 1, 2, \ldots N$

$$I_n = \sum_{i=1}^{N} \left(\frac{2m_i}{B} \right)^n . \tag{60}$$

The m_i are non-negative integers satisfying the constraint

$$m_{i+1} - m_i \leq k + 1 , \quad i = 1, 2, \ldots N - 1 . \tag{61}$$

This is very similar to the corresponding classical constraint on the eigenvalues of $w^\dagger w$ upon putting $m_i = Be_i/2$ and $k = \theta B$, the only difference being the shift of k to $k + 1$. This is the well-known level shift of the coefficient of the Chern-Simons term, leading to the "fermionization" of the matrix model and the renormalization of the filling fraction.

The spherical constraint, now, as expressed in terms of I_n can be mapped back to constraints on the m_i. Since m_i enters exactly as $Be_i/2$ in the expressions for I_n, this constraint simply reads

$$m_i \leq 2BR^2 . \tag{62}$$

One can picture the m_i as N points on a non-negative linear integer lattice, with the extra constraint (61) that they have to be at least $\ell = k + 1$ lattice units apart. The constraint (62) means that they must also be less than $2BR^2$. In fact, closer scrutiny reveals that $2BR^2$ must fall *exactly* on a lattice point. The reason is similar to the one that makes the spectrum of m_i start from zero: the point $w = 0$ corresponds to the north pole of the sphere and w must have a zero mode there to prevent the appearance of unphysical states lying "north" of the pole. Similarly, the point at $w^\dagger w = 2R$, where the spherical constraint is saturated, corresponds to the south pole of the sphere and there must be a state saturating it to prevent the creation of unphysical states "south" of the south pole. We conclude that there must be a lattice point m satisfying

$$2BR^2 = m . \tag{63}$$

Written in the form

$$B4\pi R^2 = 2\pi m \tag{64}$$

the above recovers the standard monopole quantization for the sphere, restricting the total flux to an integer number of flux quanta. The integer m is the monopole number. (The

above discussion may strike the reader as somewhat heuristic. In fact, it parallels the discussion of the corresponding scalar system of a massless particle on the sphere where the monopole quantization is also recovered. A more careful treatment would involve adding a proper quadratic kinetic term for the model as a regulator, but we will forgo any such elaboration here.)

In conclusion, we see that the quantum model contains two quantized parameters, the monopole number m and the "level" number $\ell = k + 1$. The ground state of the model corresponds to the quasi-occupation numbers having their lowest values, namely $0, \ell, 2\ell, \ldots (N-1)\ell$. For this to exist we must have

$$(N - 1)\ell \leq m . \tag{65}$$

The saturation value $(N - 1)\ell = m$ corresponds to a fully filled sphere. Given that the lowest Landau level of a monopole m magnetic field on the sphere level is in the $j = \frac{m}{2}$ angular momentum representation, it contains $L = 2j + 1 = m + 1$ states (corresponding to the lattice points $0, 1, 2, \ldots m$). The ratio N/L for the saturated state above, then, is

$$\nu = \frac{L + \ell - 1}{\ell L} . \tag{66}$$

In the limit $L, N \to \infty$ we recover a filling fraction

$$\nu = \frac{1}{\ell} = \frac{1}{k + 1} . \tag{67}$$

This confirms that the inverse filling fraction is given by the renormalized coupling constant ℓ, rather than k.

In the non-saturated case the above ground state forms a "droplet" of $\nu = 1/\ell$ quantum Hall fluid filling a northern section of the sphere. The degeneracy G of the possible Hall states is given by the number of possible distinct ways to place the N integers m_i on the lattice $0, 1, \ldots m$ respecting the constraints $m_{i+1} - m_i \geq \ell$. It can be shown that this degeneracy is

$$G = \frac{[m - (N - 1)\ell + N]!}{[m - (N - 1)\ell]! N!} . \tag{68}$$

This exactly matches the degeneracy of the corresponding $\nu = 1/\ell$ Haldane states on a sphere with a monopole of strength m at the center.[c] We have therefore established the correspondence of this model with the fractional quantum Hall states on the sphere.

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[c] For comparison, our monopole number m is Haldane's $2S$ and our level ℓ is Haldane's m.

The above analysis can be repeated in terms of the unitary model H, U. The treatment is along the same lines, leading to a similar picture for the states and the same quantization conditions for the monopole and the filling fraction.

7 Concluding remarks

We have presented a model that realizes the fractional quantum Hall system on a sphere as a matrix model, extending previous work on the plane and cylinder, and demonstrated that it reproduces the Hilbert space of particles as predicted by the standard Haldane states.

Apart from representing a technical advance in the quantum Hall matrix model technology, this also puts the issue of mapping of states and filling fraction on a firmer footing. The filling fraction in the present model is the ratio of two integers, N and $L = m + 1$, and therefore is unambiguously defined and shown to equal $1/(k + 1)$, confirming the renormalization of the level number to $k + 1$. In the planar case, in the absence of an exact operator mapping for the density, this was somewhat open to interpretation.

An interesting open issue is the derivation of the exact angular momentum operators $J_\pm, J_3$ in the quantum case. These would be given by some quantum ordering of the classical expressions (20) and would satisfy the full quantum $SU(2)$ algebra. The identification of these operators would essentially resolve the question of the mapping of Hilbert space states in the Hall system and the matrix model. Indeed, the ground state is unambiguously defined as the fully filled circular droplet, just as in the planar case, but excited states are degenerate (in terms of their energy or expectation value of x_3 coordinates) and the exact mapping is ambiguous. These states, however, group into distinct irreducible representations of the rotation algebra of the sphere and are uniquely identified by their corresponding quantum numbers. Identifying this algebra would, then, afford us the full Hilbert space map.

Several other issues remain for further investigation. The model is essentially a (gauged) matrix generalization of the corresponding scalar model of a particle on the sphere. It is known that the scalar model derives from a Kirillov action over an SU(2) group manifold, upon proper reduction to a quotient space SU(2)/U(1). There should be a corresponding Kirillov-type action for the full matrix model with interesting mathematical properties,

whose derivation remains an open issue. Further, Calogero-type models have been mapped to two-dimensional gauge theory and three-dimensional topological gauge theories [39, 40]. The above spherical model, then, should correspond to a topological field theory, as it has a finite-dimensional Hilbert space, whose derivation is another interesting problem. Finally, the physical question of properly incorporating spin and composite filling fractions in the matrix model is still an open issue.

Acknowledgments

Appendix

In this appendix we prove that the charges $J_\pm, J_3$ satisfy the $su(2)$ Lie algebra[d]

$$
\begin{cases}
i\{J_3, J_\pm\} &= \pm J_\pm \ , \\
i\{J_+, J_-\} &= 2J_3 \ .
\end{cases}
\tag{69}
$$

We shall use the parametrization of z introduced in section (5)

$$
U = (zz^\dagger)^{-\frac{1}{2}} z \ , \quad H = (zz^\dagger - 1)(zz^\dagger + 1)^{-1} \ .
\tag{70}
$$

Then the charges are given by

$$
\begin{cases}
J_3 &= BR^2 \operatorname{Tr}(H) \ , \\
J_+ &= BR^2 \operatorname{Tr}(\sqrt{1 - H^2}\, U) \ , \\
J_- &= BR^2 \operatorname{Tr}(\sqrt{1 - H^2}\, U^{-1}) \ .
\end{cases}
\tag{71}
$$

[d]More precisely, we have computed the algebra of Poisson brackets. Computing the quantum commutator of these charges is an interesting open problem.

From the kinetic term of the lagrangian (55) given by $\mathcal{L} = iBR^2\mathrm{Tr}(H\,\dot{U}U^{-1})$ we read off the Poisson brackets

$$\begin{cases} \{U_1^{-1}H_1, U_2\} & = & \frac{1}{iBR^2}T_{12} \ , \\ \{U_1, U_2\} & = & 0 \ , \\ \{U_1^{-1}H_1, U_2^{-1}H_2\} & = & 0 \ . \end{cases} \tag{72}$$

Here, as in the main text, the subscripts denote the space in which the matrices operate and $(T_{12})_{j_1 j_2}^{i_1 i_2} = \delta_{j_1}^{i_2}\delta_{j_2}^{i_1}$ is an operator exchanging the two spaces. These relations (72) can be written equivalently as

$$\begin{cases} \{H_1, H_2\} & = & \frac{1}{iBR^2}(T_{12}H_2 - T_{12}H_1) \ , \\ \{U_1, U_2\} & = & 0 \ , \\ \{H_1, U_2\} & = & \frac{1}{iBR^2}T_{12}U_2 \ . \end{cases} \tag{73}$$

Using the identity

$$\{H_1, f(H_2)\} = \frac{1}{iBR^2}[T_{12}f(H_2) - T_{12}f(H_1)] \ , \tag{74}$$

where f is an arbitrary function, we can check that

$$\{H_1, f(H_2)U_2\} = \frac{1}{iBR^2}T_{12}f(H_2)U_2 \tag{75}$$

and after tracing on both subspaces we get

$$i\{BR^2\mathrm{Tr}(H), BR^2\mathrm{Tr}(f(H)U)\} = BR^2\mathrm{Tr}(f(H)U) \ . \tag{76}$$

In the same way we can show that

$$i\{BR^2\mathrm{Tr}(H), BR^2\mathrm{Tr}(f(H)U^{-1})\} = -BR^2\mathrm{Tr}(f(H)U^{-1}) \ . \tag{77}$$

Then if we set $f(H) = \sqrt{1 - H^2}$ in (76) and (77) we obtain the first line in (69).

Deriving the last line in (69) is a little more difficult and the specific form of the function f is needed. First let us assume that f has the expansion

$$f(H) = \sum_{k=0}^{\infty} a_k H^k \ . \tag{78}$$

20

Then for an arbitrary matrix valued function A we have

$$\{A_1, f(H_2)\} = \sum_{m,n=0}^{\infty} a_{m+n-1} H_2^m \{A_1, H_2\} H_2^n \ , \tag{79}$$

where $a_{-1} = 0$ is assumed to vanish.

As above we start with $\{f(H_1)U_1, f(H_2)U_2^{-1}\}$ and after expanding using (74) and (79) and tracing we obtain

$$\{\mathrm{Tr}(f(H)U), \mathrm{Tr}(f(H)U^{-1})\} = \sum_{m,n=0}^{\infty} ia_{m+n-1} \mathrm{Tr}(f(H)H^m U^{-1} H^n U + f(H)H^m U H^n U^{-1}) \ .$$

After further expanding f, changing the order of summation and some index redefinition, we obtain

$$\{\mathrm{Tr}(f(H)U), \mathrm{Tr}(f(H)U^{-1})\} = \frac{i}{BR^2} \sum_{m,n=0}^{\infty} \alpha_{m+n+1} \mathrm{Tr}(H^m U^{-1} H^n U) \ , \tag{80}$$

where $\alpha_t = \sum_{k=0}^{t} a_{t-k} a_k$. The convolution coefficients can be recognized as the expansion coefficients of $f^2(H) = 1 - H^2$ so only $\alpha_0 = 1$ and $\alpha_2 = -1$ are nonvanishing and furthermore only α_2 contributes to the sum. Finally we obtain

$$i\{BR^2 \mathrm{Tr}(\sqrt{1 - H^2})U), BR^2 \mathrm{Tr}(\sqrt{1 - H^2}U^{-1})\} = 2BR^2 \mathrm{Tr}(H) \ . \tag{81}$$

This completes the proof of the relations (69). In the derivation it was essential to be able to expand the function $f(H)$ as in (78). This is always possible in our case since all the eigenvalues of H have absolute values less than one. The $SO(3)$ symmetry is not a symmetry of the general unitary model and the restrictions on the integrals of motions are essential.

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